

# A SETTING FOR BRANCHED CONFIGURATIONS

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**ABSTRACT.** We describe a setting that enlarges the notion of finite or countable configurations of a manifold which is a candidate to the description of branching processes and turbulence effects in 1-dimensional dynamics. This is a modified structure (topological and differential) on the finite configuration space [4] that fits with the announced settings. The branching condition, which can be understood as a “conservation of global velocity”, appears as a consequence of smoothness for vague convergence. We finish with remarks for future developments.

## 1. BRANCHED CONFIGURATION SPACES

We first recall the definition of configuration spaces

**1.1. Configuration spaces.** Let us describe step by step a way to build infinite configurations. As an example, we explain each step with the configurations already defined in e.g. [1] and [4]. A set of 1-configurations is a set  $\Gamma^1$  of objects that are modelizations of physical quantities. For example, in the setting [1], [5], and in the setting [4], the physical quantity modeled is the position of one particle. The whole world is modeled as a locally compact manifold  $M$ , and the set of 1-configurations is itself  $M$ , or equivalently the set of all Dirac measures on  $M$ .

Let  $I$  be a set of indexes.  $I$  can be countable or uncountable. We define the **indexed** (or the **ordered** if  $I \subset \mathbb{N}$  equipped with its total order) configuration spaces. For this, we need to define a symmetric binary relation  $\mathcal{U}$  on  $\Gamma^1$ , that express the compatibility of two physical quantities. We assume also that  $\mathcal{U}$  has the following property:

$$(1.1) \quad \forall (u, v) \in (\Gamma^1)^2, \quad u\mathcal{U}v \Rightarrow u \neq v.$$

In the settings [5], [1] and [4], two particles cannot have the same position. Then, for  $x, y \in N$ ,

$$x\mathcal{U}y \iff x \neq y.$$

With these restrictions, we can define the indexed or ordered configuration spaces :

$$\begin{aligned} O\Gamma^n &= \{(u_1, \dots, u_n) \in (\Gamma^1)^n \text{ such that, if } i \neq j, \quad u_i\mathcal{U}u_j\} \\ O\Gamma &= \coprod_{n \in \mathbb{N}^*} O\Gamma^n \text{ and} \\ O\Gamma^I &= \{(u_n)_{n \in I} \in (\Gamma^1)^I \text{ such that, if } i \neq j, \quad u_i\mathcal{U}u_j\}. \end{aligned}$$

The general configuration spaces are not ordered. Let  $\Sigma_n$  be the group of bijections on  $\mathbb{N}_n$ , and  $\Sigma_I$  be the set of bijections on  $I$ . We can define two actions:

$$\begin{aligned}\Sigma_n \times O\Gamma^n &\rightarrow O\Gamma^n \\ (\sigma, (u_1, \dots, u_n)) &\mapsto (u_{\sigma(1)}, \dots, u_{\sigma(n)})\end{aligned}$$

and its infinite analog:

$$\begin{aligned}\Sigma_I \times O\Gamma^I &\rightarrow O\Gamma^I \\ (\sigma, (u_n)_{n \in I}) &\mapsto (u_{\sigma(n)})_{n \in I}\end{aligned}$$

where  $\Sigma_I$  is a subgroup of the group of bijections of  $I$ . In the sequel,  $I$  is countable with discrete topology, which avoids topological problems on  $\Sigma_I$  as in more complex examples. Then, we define general configuration spaces:

$$\begin{aligned}\Gamma^n &= O\Gamma^n / \Sigma_n \\ \Gamma &= \coprod_{n \in \mathbb{N}^*} \Gamma^n \\ \Gamma^I &= O\Gamma^I / \Sigma_I\end{aligned}$$

**1.2. Branched configurations.** As we can see, finite configurations  $\Gamma$  are made of a countable disjoint union. We now fix a metric  $d$  on  $M$ . The idea of branched configurations is to glue together the components  $\Gamma^n$  on the generalized diagonal, namely, we define the following distance on  $\Gamma$ :

**Definition 1.1.** Let  $(u, v) \in \Gamma^2$ .

$$d_\Gamma(u, v) = \sup_{(x, x') \in u \times v} \{d(x, v), d(x', u)\}$$

**Proposition 1.2.**  $d_\Gamma$  is a distance on  $\Gamma$ .

**Proof.** We remark that  $d_\Gamma$  is the Hausdorff distance restricted to  $\Gamma$ .  $\square$

Now, we remark that  $f \in C^\infty(M, \mathbb{R})$  induces a map on  $\Gamma$  by

$$f(u) = \frac{1}{|u|} \sum_{x \in u} f(x).$$

Let  $\mathcal{F}_0 = C^\infty(M, \mathbb{R})$ . Following [7], the set of paths  $\mathcal{C} = \{c \in C^0(\mathbb{R}, \Gamma) \mid \forall f \in \mathcal{F}_0, f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\}$  defines a Frölicher space ([3], see e.g. [7]). In the sequel, we shall use the stript  $\mathcal{B}$  when  $\Gamma$  is equipped with the metric  $d_\Gamma$  and the corresponding Frölicher structure.

## 2. BRANCHED SECTIONS OF A FIBER BUNDLE

Let  $\pi : F \rightarrow M$  be a fiber bundle of typical fiber  $F_0$ . Here,  $n \in \mathbb{N}^* \cup \infty$ . Let  $\pi : F \rightarrow M$  be a fiber bundle over  $M$  with typical fiber  $F_0$ . Let

$$\Gamma_M^n(F) = \{u \in \Gamma^n(F) \mid |\pi(u)| = 1\}.$$

This is trivially a fiber bundle of basis  $M$  with typical fiber  $\Gamma^n(F_0)$ . We define also  $\Gamma_M(F) = \coprod_{n \in \mathbb{N}^*} \Gamma_M^n(F)$ , and also  $\Gamma_M^I(F)$  with the trivially adapted notations.

Let us now consider the Frölicher structure described on section 1.2. It is based on the natural diffeology carried by each  $\Gamma^n(F_0)$  ( $n \in \mathbb{N}^*$ ) and by the set of paths  $\mathcal{P}'_1$  that are paths  $\gamma : \mathbb{R} \rightarrow \Gamma(F_0)$  such that  $\exists(m, n) \in (\mathbb{N}^*)^2$ ,

- $\gamma|_{]-\infty;0]}$  is a smooth path on  $\Gamma^m(F)$ ,
- $\gamma|_{]0;+\infty[}$  is a smooth path on  $\Gamma^n(F)$ ,
- Let  $l \in \gamma(0)$ . Then the sum of infinite jets of the trajectories going to  $l$  in  $0^-$  equals to the sum of the infinite jets of the trajectories coming from  $l$  in  $0^+$ .

Remark that the last condition comes from the smoothness required for each map  $f \circ \gamma$ , with  $f \in C^\infty(F_0, \mathbb{R})$ . Then, a **finitely branched section** of  $F$  is a smooth section of  $\Gamma_M(F)$ , where “smooth” means “smooth with respect to the Frölicher structure of branched configurations of  $F_0$ ”. The first examples that come to our mind are the well-known branched processes, and we can wonder some deterministic analogues replacing stochastic processes by dynamical systems. Let us here sketch a toy example extracted from the theory of turbulence:

**Example: equilibrium of mayflies population** Assuming that Mayflies live and die in the same portion of river, the population  $p_{n+1}$  at the year  $(n+1)$  is obtained from the population  $p_n$  at the year  $n$  (after normalization procedure) by the formula

$$p_{n+1} = Ap_n(1 - p_n)$$

where  $A \in [0; 4]$  is a constant coming from the environmental data. For  $A$  small enough, the fixed point of the so-called “logistic map”  $\phi_A(x) = Ax(1 - x)$  is stable, hence the population  $p_n$  tends to stabilize around this value. But when  $A$  is increasing, the fixed point becomes unstable and  $p_n$  tends to stabilize around  $2^k$  multiple values which are the stable fixed points of the map  $\phi^{2^k}$  obtained by composition rule.

Now, assume that we consider a river (or a lake), modeled by an interval (or an open subset of  $\mathbb{R}^2$ ) that we denote by  $U$ , where the parameter  $A$  is a smooth map  $U \rightarrow [0; 4]$ . The parameter  $A$  is a smooth map  $U \rightarrow [0; 4]$  and the cardinality of configuration of equilibrium depends on the value of  $A$ .

### 3. FINAL REMARKS

*Remark 3.1.* The proposed model is made of massless particles. A generalization to massive particles is straightforward, replacing

$$u \in \Gamma$$

by a “massive configuration”

$$\tilde{u} = \left\{ (x, m_x) \mid x \in u, \sum_{x \in u} m_x = 1 \right\}$$

and setting, for  $f \in C^\infty(M, \mathbb{R})$ ,

$$f(\tilde{u}) = \sum_{x \in u} m_x f(x).$$

*Remark 3.2.* Integration of dynamical systems generates monodromy problems, coming from the fundamental group of the state space. The fundamental group and the classical topological invariants of  $\Gamma$  are partially described in [4]. What about  $\mathcal{B}\Gamma$ ? For this, we can consider either  $\mathcal{B}\Gamma$  as  $\Gamma$  “glued over” the generalized diagonal (and use Van Kampen type spectral sequences) or embed  $\mathcal{B}\Gamma$  into some more general space, e.g.  $M^{\mathbb{N}}$ . Each approach seems promising but technically difficult, and we leave these questions open (this is a work in progress).

*Remark 3.3.* In the same spirit as the previous remark, the same questions hold for the group of diffeomorphisms  $\text{Diff}(\mathcal{B}\Gamma)$ . Which are its remarkable subgroups? cycles? We already have  $\text{Diff}(M) \subset \text{Diff}(\mathcal{B}\Gamma(M))$ . What about the topological properties of this inclusion?

*Remark 3.4.* Since turbulence is a transition to chaos, a naïve configuration space for chaotic dynamical systems would be the  $d_\Gamma$ — completion of  $\mathcal{B}\Gamma$ . The properties of this (speculative) configuration space are quite exciting goals, in order to make a stronger link with metric geometry properties of the space of metric subspaces of  $M$  equipped with the Hausdorff metric.

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